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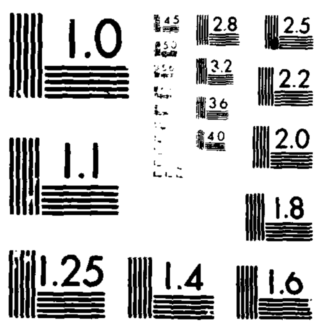
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POSITIVE DEPENDENCE IN MULTIVARIATE DISTRIBUTIONS

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POSITIVE DEPENDENCE IN MULTIVARIATE DISTRIBUTIONS

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ABSTRACT

This paper gives some new results on positive dependence between random variables which are jointly normally distributed with special reference to certain inequalities of the form $P(\underline{X}\varepsilon A, \underline{Y}\varepsilon B) \geq P(\underline{X}\varepsilon A)P(\underline{Y}\varepsilon B)$, where A and B are given sets and $\underline{X}$ and $\underline{Y}$ are random vectors. Some results are also given on statistical dependence between quadratic forms.

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1. Introduction

Let $\underline{X}$ be a random vector and let $\underline{X}_1, \ldots, \underline{X}_k$ be its partition into k sub-vectors. Let p_t denote the dimension of $\underline{X}_t$, $t=1,\ldots,k$, and let E_t be a subset of R^{p_t}, where $R = (-\infty,\infty)$. The statistical dependence between the sub-vectors, given by the inequality

$$(1.1) \qquad P\{\underline{X}_t \in E_t,\ t=1,\ldots,k\} \geq \prod_{t=1}^{k} P(\underline{X}_t \in E_t)$$

is of interest in certain problems, such as, life testing and interval estimation. The above inequality has been considered by Dunn (1958), Dykstra (1979), Khatri (1967, 1970), Sidak (1967, 1968, 1975), Scott (1967), Slepian (1962) and Tong (1970), for the case in which the distribution of $\underline{X}$ is multivariate normal. In this paper we give some new results on the inequality.

Let the distribution of $\underline{X}$ be multivariate normal with mean vector $\underline{0}$ and covariance matrix $\Sigma = (\sigma_{ij})$, and let $p_t = 1$, $t=1,\ldots,k$. If each E_t is an interval symmetric about the origin, then the inequality (1.1) follows from Theorem 1 of Sidak (1968). For the one sided case, that is, when each E_t is an interval of the form $(-\infty,a_t)$ or each E_t is of the form (a_t,∞), then the inequality holds if $\sigma_{ij} \geq 0$ $(i \neq j)$ by a result due to Slepian (1962). The reverse inequality holds if $\sigma_{ij} \leq 0$ $(i \neq j)$.

Let (Σ_{ij}) be the partition of Σ, where Σ_{ij} denotes the covariance between $\underline{X}_i$ and $\underline{X}_j$. Let $k = 2$ and let r denote the rank of Σ_{12}. The inequality (1.1) follows from Theorem 1 of Khatri (1967) if $r = 1$ and E_1 and E_2 are convex sets, symmetric about the origin. In a subsequent paper, Khatri (1970) claimed that the inequality holds also for $r > 1$. However, the proof of the generalization given by the author was found incorrect by Sidak (1975). On the other hand, Dykstra (1979) showed that the inequality holds for $r \geq 1$ if E_1 is a convex symmetric set and E_2 is the interior of an ellipsoid given by $\underline{X}_2' A \underline{X}_2 \leq c$, where $A\Sigma_{22}$ is an idempotent matrix. In this paper, we generalize the result of Dykstra. Moreover, we show that the inequality (1.1) holds for $r \geq 1$ if Σ satisfies a given condition and E_1 and E_2 are both increasing (decreasing) sets, where a set E is said to be an increasing (decreasing) set if $\underline{X} \in E$ and $\underline{y} \succ (\prec) \underline{x}$ then $\underline{y} \in E$, where $\succ (\prec)$ means $\geq$ ($\leq$) component wise.

Very few results are known for the case in which the underlying distribution is not assumed to be multivariate normal. A set of random variables $Y_1,\ldots,Y_n$ are said to be associated if the covariance between $f(Y_1,\ldots,Y_n)$ and $g(Y_1,\ldots,Y_n)$ is non-negative for all functions f and g which are nondecreasing in each argument. The variables $Y_1,\ldots,Y_n$ are said to be positively orthant dependent if the following inequality holds for all values a_i.

$$(1.2) \qquad P(Y_i \leq a_i,\ i=1,\ldots,n) \geq \prod_{i=1}^{n} P(Y_i \leq a_i).$$

The above inequality holds if $Y_1,\ldots,Y_n$ are associated by Theorem 5.1 of Esary, Proschan and Walkup (1967). In Section 3 we show statistical dependence, including the association between certain quadratic forms, and in Section 4 we introduce the concept of negative association.

2. Probability Inequalities

Consider the inequality (1.1) for k=2. Let $\underline{X}_1, \underline{X}_2$ be jointly normally distributed with mean vector $\underline{0}$ and covariance matrix Σ Let E_1 be a convex set symmetric about the origin and let E_2 denote the interior of the ellipsoid given by $\underline{X}_2' A \underline{X}_2 \leq c$, where A is a positive semi-definite matrix. We shall show that the inequality (1.1) holds under certain conditions on A and Σ. First we give a lemma whose proof is given at the end of the section. The lemma is required in the proof of Theorems 2.1 and 2.2 below.

Lemma 2.1 Given $s \geq r = \text{rank}(\Sigma_{12})$, there exist matrices Σ_{13} and Σ_{32} of order $p_1$xs and sxp_2,respectively,such that $\Sigma_{12}=\Sigma_{13}\Sigma_{32}$. Moreover, $\Sigma_{11} - \Sigma_{13}\Sigma_{31}$ and $\Sigma_{22} - \Sigma_{23}\Sigma_{32}$ are positive definite, where $\Sigma_{31} = \Sigma_{13}'$ and $\Sigma_{23} = \Sigma_{32}'$.

Let $SS' = \Sigma_{11} - \Sigma_{13}\Sigma_{31}$ and $TT' = \Sigma_{22} - \Sigma_{23}\Sigma_{32}$, where Σ_{13} and Σ_{23} are constructed form Lemma 2.1. Then the joint distribusion of $\underline{X}_1$, $\underline{X}_2$ can be represented as follows:

$$\begin{aligned} \underline{X}_1 &= S\underline{U} + \Sigma_{13}\,\underline{W} \\ \underline{X}_2 &= T\underline{V} + \Sigma_{23}\,\underline{W} \end{aligned} \qquad (2.1)$$

where $\underline{U}$, $\underline{V}$ and $\underline{W}$ are normally distributed independent random vectors, $\underline{U} \overset{d}{\sim} N(\underline{0}, I_{p_1})$, $\underline{V} \overset{d}{\sim} N(\underline{0}, I_{p_2})$ and $\underline{W} \overset{d}{\sim} N(\underline{0}, I_s)$. Clearly, $\underline{X}_1$, $\underline{X}_2$ are conditionally independent given $\underline{W}$.

Theorem 2.1. The inequality (1.1) holds for k=2 if (i) $A(\Sigma_{22} - \Sigma_{23}\Sigma_{32})$ is an idempotent matrix and (ii) $\Sigma_{32}A\Sigma_{23} = \eta I_s$ for some positive number η, where Σ_{23} is constructed from Lemma 2.1.

Proof. Let $Q = \underline{W}'\underline{W}$ and $\underline{Y} = \Sigma_{13}\, \underline{W}\, Q^{-\frac{1}{2}}$. From the sufficiency, completeness and invariance considerations we find that $\underline{Y}$ is independent of Q. From the representation (2.1) we have that given $\underline{W}_1$, $\underline{X}_1$ is normally distributed with mean vector $\underline{Y}\, Q^{\frac{1}{2}}$ and $\underline{X}_2'A\underline{X}_2$ is distributed according to a noncentral chi-square distribution with noncentrality parameter equal to ηQ. Thus

$$(2.2) \qquad P(\underline{X}_1 \varepsilon E_1, \underline{X}_2 \varepsilon E_2) = E_{\underline{W}} P(\underline{X}_1 \varepsilon E_1 | \underline{W})\; P(\underline{X}_2 \varepsilon E_2 | \underline{W})$$

$$= E_Q[P(\underline{X}_2 \varepsilon E_2 | Q)\; E_{\underline{Y}} P(\underline{X}_1 \varepsilon E_1 | \underline{Y}, Q)] \; .$$

Clearly, $P(\underline{X}_2 \varepsilon E_2 | Q)$ is a decreasing function of Q. Moreover $E_{\underline{Y}}(\underline{X}_1 \varepsilon E_1 | \underline{Y}, Q)$ is a non increasing function of Q by Theorem 1 of Anderson (1955). Therefore by Kimball's inequality the quantity on the right side of (2.2) is greater than or equal to

$$E_Q P(\underline{X}_2 \varepsilon E_2 | Q) \cdot \; E_{Q,\underline{Y}} P(\underline{X}_1 \varepsilon E_1 | \underline{Y}, Q) = P(\underline{X} \varepsilon E_1)\; P(\underline{X}_2 \varepsilon E_2) \; .$$

The theorem is proved.

Corollary 2.1. The inequality (1.1) holds for k=2 if $A = \Sigma_{22} = I_{p_2}$.

Proof: Let ε be a small positive number such that $\Sigma_{11} - (\Sigma_{12}\Sigma_{21})/(1-\varepsilon)$ is a positive definite matrix. Then $\Sigma_{13} = (1-\varepsilon)^{-\frac{1}{2}}\, \Sigma_{12}$ and $\Sigma_{23} = (1-\varepsilon)^{\frac{1}{2}} I_{p_2}$ satisfy the conditions of Lemma 2.1. It is sufficient to prove the corollary for $A = \varepsilon^{-1} I_{p_2}$. Now the conditions of Theorem 2.1

are satisfied for $A = \varepsilon^{-1} I_{p_2}, \Sigma_{22} = I_{p_2}$ and $\eta = (1-\varepsilon)/\varepsilon$. Hence the inequality (1.1) holds. A slightly different proof of the above corollary has been given by Dykstra (1979).

Corollary 2.2. The inequality (1.1) holds for k=2, if $A\Sigma_{22}$ is an idempotent matrix.

The proof of Corollary 2.2 is omitted.

There is a large class of matrices Σ and A for which Theorem 2.1 is applicable. An example is given here for illustration:

Let

$$\Sigma_{11} = \begin{pmatrix} 25 & -4 & 0 \\ -4 & 25 & 12 \\ 0 & 12 & 25 \end{pmatrix}, \Sigma_{22} = \begin{pmatrix} 6 & 2 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 2 & 6 \end{pmatrix}, \Sigma_{12} = \begin{pmatrix} 0 & 0 & 4 & 4 \\ 6 & 6 & -2 & -2 \\ 4 & 4 & 0 & 0 \end{pmatrix}$$

and $A = \frac{1}{4} I_4$, $\Sigma_{13} = \begin{pmatrix} 2 & -2 \\ 2 & 4 \\ 2 & 2 \end{pmatrix}$, $\Sigma_{23} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$.

Note that Corollary 2.2 is not applicable since $A\Sigma_{22}$ is not an idempotent matrix.

Next, we consider the case in which E_1 and E_2 are both decreasing or both increasing sets. A matrix is called nonnegative if its elements are nonnegative. Let Σ_{13} and Σ_{23} be constructed from Lemma 2.1.

Theorem 2.2. The inequality (1.1) holds for k=2, if Σ_{13} and Σ_{23} are nonnegative matrices and E_1 and E_2 are decreasing (increasing) sets.

Proof: From the representation (2.1) we have

$$(2.3) \qquad P(\underline{X}_1 \varepsilon E_1, \underline{X}_2 \varepsilon E_2) = E_{\underline{W}} P(\underline{X}_1 \varepsilon E_1 | \underline{W}) P(\underline{X}_2 \varepsilon E_2 | \underline{W}).$$

Let E_1 and E_2 be decreasing sets. Since E_1 is a decreasing set and $E\underline{X}_1|\underline{W} = \Sigma_{13}\underline{W}$, where Σ_{13} is a nonnegative matrix, it follows that $P(\underline{X}_1 \varepsilon E_1|\underline{W})$ is non-increasing in each component of $\underline{W}$. Similarly $P(\underline{X}_2 \varepsilon E_2|\underline{W})$ is non-increasing in each component of $\underline{W}$. As the components of $\underline{W}$, being independent, are associated we have

$$(2.4) \quad E_{\underline{W}}P(\underline{X}_1 \varepsilon E_1|\underline{W})P(\underline{X}_2 \varepsilon E_2|\underline{W}) \geq E_{\underline{W}}P(\underline{X}_1 \varepsilon E_1|\underline{W})E_{\underline{W}}P(\underline{X}_2 \varepsilon E_2|\underline{W})$$

$$= P(\underline{X}_1 \varepsilon E_1)P(\underline{X}_2 \varepsilon E_2) \; .$$

The conclusion of the theorem follows from (2.3) and (2.4). The above result is proved similarly for the case in which E_1 and E_2 are increasing sets.

Corollary 2.3. The inequality (1.1) holds for k=2 if Σ_{12} is a nonnegative matrix of rank 1.

Proof: Let Σ_{12} can be written as $\underline{\lambda}\,\underline{\mu}'$ where $\underline{\lambda}$ is a p_1x1 vector and $\underline{\mu}$ is a p_2x1 vector. Since $\lambda_i\mu_j \geq 0$ for all i and j, all non-zero components of $\underline{\lambda}$ and $\underline{\mu}$ have the same sign. So it may be assumed that $\underline{\lambda}$ and $\underline{\mu}$ are nonnegative vectors. The conditions of Theorem 2.2 are satisfied by letting $\Sigma_{13} = \underline{\lambda}$ and $\Sigma_{23} = \underline{\mu}$.

Corollary 2.4. Let $X_1,\ldots,X_k$ be jointly normally distributed and let their covariance matrix be nonnegative. Then $X_1,\ldots,X_k$ are positively orthant dependent.

The above corollary follows directly from Corollary 2.3. It is a special case of a more general result due to Slepian (1962). The proof is omitted.

A repeated application of Corollary 2.3 shows that the inequality (1.1) holds for $k \geq 2$ in the following cases,where the covariance matrix Σ is given by (i) $\sigma_{ii} = 1$, $\sigma_{ij'} = \rho > 0$, $i \neq j$, (ii) $\sigma_{ij} = \rho^{|i-j|}$ for all i and j, (iii) $\sigma_{ii} = 1$, $\rho_{ij} = \rho > 0$ for $|i-j| = 1$, zero otherwise.

A matrix A is said to be completely positive if $A = PP'$ where P is a nonnegative matrix. Let A be of order $n \times n$ and let $A(i, s,\ldots,n|j,s,\ldots,n)$ denote the minor of A with rows indexed by $i,s,\ldots,n$ and columns indexed by $j,s,\ldots,n$. A sufficient condition for a nonnegative positive definite matrix to be completely positive is that $A(i,s,\ldots,n|j,s,\ldots,n) > 0$ for $1 \leq i, j \leq s$ and $2 \leq s \leq n$ by Corollary 5.1 of Markham (1971).

<u>Corollary 2.5</u>. The inequality (1.1) holds if $\Sigma(i,s,\ldots,n|j,s,\ldots,n)>0$ for $1 \leq i, j < s$, and $2 \leq s \leq n$, where n denotes the order of Σ.

Proof: Let $k = 2$. The conditions of the corollary imply that Σ is completely positive and so also is $\Sigma-\varepsilon I_n$ for sufficiently small $\varepsilon > 0$. Therefore let $\Sigma - \varepsilon I_n = PP'$, where P is nonnegative. Let P_1, P_2 be a partition of P where P_1 is of order $p_1 x n$ and P_2 is of order $p_2 x n$. Then $\Sigma_{12} = P_1 P_2'$, $\Sigma_{11} - \varepsilon I_{p_1} = PP_1'$, $\Sigma_{22} - \varepsilon I p_2 = P_2 P_2'$. Let $\Sigma_{13} = P_1$ and $\Sigma_{23} = P_2$. Then $\Sigma_{12} = \Sigma_{13}\Sigma_{32}$. Moreover $\Sigma_{11} - \Sigma_{13}\Sigma_{31}$ and $\Sigma_{22} - \Sigma_{23}\Sigma_{32}$ are positive definite.Then the inequality (1.1) follows from the application of Theorem 2.2. A repeated application of the above proof yields the inequality for $k \geq 2$.

<u>Proof of Lemma 2.1</u>. Let $\lambda_1,\ldots,\lambda_r$ denote the positive characteristic roots of $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}$. Since Σ is positive definite, it follows that $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}{}_{21}$ is positive definite. Therefore $0 < \lambda_i < 1$ for $i = 1,\ldots,r$.

Let P_1 and P_2 be non-singular matrices such that $P_1'\Sigma_{11}P_1 = I_m$ and $P_2'\Sigma_{22}P_2 = I_n$. By the singular value decomposition theorem there exist orthogonal matrices Q_1 and Q_2, such that

$$Q_1'P_1'\Sigma_{12}P_2Q_2 = D$$

where $D = (d_{ij})$ is given by $d_{ij} = 0$, $i \neq j$, $d_{ii} = \lambda_i^{1/2}$ for $i=1,\ldots,r$ and $d_{ii} = 0$ for $i > r$, $i=1,\ldots,m$ and $j=1,\ldots,n$. Let $A_{m,s} = (a_{ij})$ denote an mxs matrix in which $a_{ij} = 0$ for $i \neq j$, $a_{ii} = \lambda_i^{1/4}$ $i=1,\ldots,r$, $a_{ii} = 0$ for $i > r$, $i=1,\ldots,m$, $j=1,\ldots,s$. Now let

$$\Sigma_{13} = (P_1^{-1})'Q_1A_{m,s} \quad \text{and} \quad \Sigma_{23} = (P_2^{-1})'Q_2A_{n,s}$$

since $A_{m,s}A_{n,s}' = D$, we have that $\Sigma_{12} = \Sigma_{13}\Sigma_{31}$. Furthermore the matrix $\Sigma_{11} - \Sigma_{13}\Sigma_{31}$ is positive definite since it can be written as

$$\Sigma_{11} - \Sigma_{13}\Sigma_{31} = (P_1^{-1})'Q_1CQ_1'P_1^{-1}$$

where C is an mxm diagonal matrix, the diagonal elements being $1-\lambda_1^{1/2},\ldots,1-\lambda_r^{1/2},1,\ldots,1$. Similarly the matrix $\Sigma_{22}-\Sigma_{23}\Sigma_{32}$ is positive definite.

3. Association of Quadratic Forms

Let $\underline{Y}$ be normally distributed with mean vector $\underline{0}$ and covariance matrix Σ . In this section we give some results on the statistical dependence among the quadratic forms $\underline{Y}' A_i \underline{Y}$, $i=1,\ldots,k$, where A_i are positive semidefinite matrices.

Theorem 3.1. The correlation between two quadratic forms $\underline{Y}'A_i\underline{Y}$ and $\underline{Y}'A_j\underline{Y}$ is nonnegative.

Proof: By direct computation we have

$$\text{(3.1)} \qquad \operatorname{cov}(\underline{Y}'A_i\underline{Y}, \underline{Y}'A_j\underline{Y}) = 2 \text{ trace } A_i \Sigma A_j \Sigma.$$

Since A_i and A_j are positive semidefinite, the right side of (3.1) is nonnegative by Theorem 9.1.28 of Graybill (1969).

Remark. If $A_i \Sigma A_j$ is positive semidefinite then it can be shown that the conclusion of the theorem holds without the assumption that $E\underline{Y} = \underline{0}$.

Association between two random variables implies that the correlation between the variables is nonnegative. Theorems 3.2 through 3.4 below, give certain conditions for the association of quadratic forms. We say that the matrices A_i are diagonalized simultaneously through a non-singular transformation if there exists a non-singular matrix P such that $PA_iP'=D_i$, where D_i is a diagonal matrix, $i=1,\ldots,k$. Clearly the diagonal elements of each D_i are nonnegative. If for example, $A_iA_j = A_jA_i$ for each pair (i,j), then the matrices A_i are diagonalized simultaneously through an orthogonal transformation.

Theorem 3.2. Let $n = k = 2$. If A_1 and A_2 can be diagonalized simulatenously then $\underline{Y}'A_1\underline{Y}$ and $\underline{Y}'A_k\underline{Y}$ are associated.

Proof: Under the conditions of the theorem we can assume without loss of generality that A_1 and A_2 are diagonal matrices. Also, Σ can be assumed to be a correlation matrix. Let ρ denote the correlation between Y_1 and Y_2. Now, Y_1^2 is conditionally distributed, given Y_2^2, as $(1-\rho^2)\chi^2_{1,\delta}$, a non-central chi-square with 1 degree of freedom and non-centrality parameter $\delta = \rho^2 Y_2^2 / (1-\rho^2)$. Thus the conditional distribution of Y_1^2 given Y_2^2 is stochastically increasing in Y_2^2. Therefore, Y_1^2 and Y_2^2 are associated by theorem 4.7 of Barlow and Proschan (1975). The theorem follows, since $\underline{Y}'A_1\underline{Y}$ and $\underline{Y}'A_2\underline{Y}$ are non-decreasing functions of Y_1^2 and Y_2^2.

Suppose that the components of $\underline{Y}$ can be grouped in pairs, that is $\underline{Y} = (\underline{X}_1', \ldots, \underline{X}_k')'$, where each $\underline{X}_i$ is a two-component vector. Let each A_i be of order 2×2. The proof of theorem 3.1 below is straightforward.

Theorem 3.3. If $\underline{X}_i$ and $\underline{X}_j$ are uncorrelated $(i \neq j)$, and $A_1, \ldots, A_k$ can be diagonalized simultaneously then $\underline{X}_1'A_1\underline{X}_1, \ldots, \underline{X}_k'A_k\underline{X}_k$ are associated.

Suppose that the matrices A_i commute pairwise with respect to Σ, that is $A_i \Sigma A_j = A_j \Sigma A_i$ $(i \neq j)$. Then through a non-singular transformation we can write $\underline{Y}'A_i\underline{Y} = \sum_{j=1}^{n} a_{ij} Z_j^2$, $i=1,\ldots,k$, where $a_{ij} \geq 0$ and $Z_1, \ldots, Z_n$ are independent normal random variables. Therefore, we have the following result.

Theorem 3.4. If $A_1, \ldots, A_k$ commute pair wise with respect to Σ then $\underline{Y}'A_1\underline{Y}, \ldots, \underline{Y}'A_k\underline{Y}$ are associated.

The property of association of random variables is observed in many statistical models. Consider, for example, the model

$$Y_i = \rho Y_{i-1} + \varepsilon_i \quad , \quad i=1,2,\ldots.$$

for a time series, where ε_i are independent random variables. The distribution of Y_i given $Y_1,\ldots,Y_{i-1}$ depends on Y_{i-1} only, by the Markov property. If $\rho \geq 0$ then the conditional distribution is stochastically increasing in sequence. On the other hand, if the distribution of each ε_i is unimodal and symmetric about the origin then the conditional distribution of Y_i^2 given $Y_1^2,\ldots,Y_{i-1}^2$ is stochastically increasing in Y_{i-1}^2 and thus the joint distribution of $Y_1^2,\ldots,Y_n^2$ is stochastically increasing in sequence. It follows from theorem 4.7 of Barlow and Proschan (1975) that $Y_1,\ldots,Y_n$ are associated if $\rho \geq 0$, and that $Y_1^2,\ldots,Y_n^2$ are associated if the distribution of each ε_i is unimodal and symmetric about the origin.

4. Negative Association

The definition of association between a set of random variables, given above, implies positive dependence in the sense that large (small) values of any subset of the given variables are associated with large (small) values of any other subset of the variables. A weaker form of dependence and of the opposite kind is implied by the definition of negative association, given below. The random variables $X_1,\ldots,X_k$ $(k\geq 2)$ are negatively associated if the covariance, when it exists, between $f(\underline{Y})$ and $g(\underline{Z})$ is non-negative for all nondecreasing functions f and g, where $\underline{Y}$ and $\underline{Z}$ are subvectors representing a partition of the given variables into two subsets. Two examples of negatively associated random variables are given below.

Example 4.1. Let $\underline{X} = (X_1,\ldots,X_k)$ be distributed according to the multivariate normal distribution with mean vector $\underline{0}$ and covariance $\Sigma = (\sigma_{ij})$, where $\sigma_{ii} = 1$, $\sigma_{ij} = -\rho$ $(i \neq j)$ and $0 < \rho < (k-1)^{-1}$. Let the subvectors $\underline{Y}$ and $\underline{Z}$ be obtained through a partition of the components of $\underline{X}$. As the mean of the conditional distribution of $\underline{Y}$ given $\underline{Z}$ is for each component, a linear function of the components of $\underline{Z}$ with negative coefficients, the conditional distribution of $\underline{Y}$ given $\underline{Z}$ is stochastically decreasing in each component of $\underline{Z}$. Therefore, the conditional expection of $f(\underline{Y})$ given $\underline{Z}$ is nondecreasing in the components of $\underline{Z}$ for any nondecreasing function f. An application of Kimball's inequality gives

$$E\, f(\underline{Y})g(\underline{Z}) \leq E\, f(\underline{Y})\; E\, g(\underline{Z}).$$

Therefore, the random variables $X_1,\ldots,X_k$ are negatively associated.

<u>Example 4.2</u>. Let the random variables $X_1,\ldots,X_k$ be jointly distributed according to the Dirichlet distribution, given by the density function

$$f(x_1,\ldots,x_k) = \frac{\Gamma(\nu_1+\ldots+\nu_k)}{\Gamma(\nu_1)\ldots\Gamma(\nu_k)}\; x_1^{\nu_1-1}\ldots x_k^{\nu_k-1}$$

where $x_i > 0$ $(i=1,\ldots,k)$, $\sum_{i=1}^{k} x_i = 1$. The Dirichlet distribution can be represented as follows:

$$X_i = V_i\,/\,S\;,\quad i=1,\ldots,k$$

where $V_1,\ldots,V_k$ are independent gamma random variables with $\nu_1,\ldots,\nu_k$ degrees of freedom, respectively, and $S = \sum_{i=1}^{k} V_i$. From the above representation it follows that the conditional distribution of $\underline{Y}$ given $\underline{Z}$ is stochastically decreasing in $\underline{Z}$, where $\underline{Y}$ and $\underline{Z}$ are defined as in the previous example. Therefore, the random variables $X_1,\ldots,X_k$ are negatively associated. The Dirichlet distribution arises in goodness of fit tests based on sample spacings (see e.g. Pyke (1965)).

It is easy to show that the following sets of random variables are negatively associated: (i) independent random variables, (ii) nondecreasing functions $g_i(X_i)$ of negatively associated random variables X_i's, (iii) union of independent sets of negatively associated random variables. Also, it follows directly from the definition of negative association that the reverse inequality holds in (1.1) for k = 2 if E_1 and E_2 are nondecreasing sets and the components of $\underline{X}$ are negatively associated.

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This paper gives some new results on positive dependence between random variables which are jointly normally distributed with special reference to certain inequalities of the form $P(\underline{X}\in A, \underline{Y}\in B) \geq P(\underline{X}\in A)P(\underline{Y}\in B)$, where A and B are given sets and $\underline{X}$ and $\underline{Y}$ are random vectors. Some results are also given on statistical dependence between quadratic forms.

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